Final exam for Kwantumfysica 1 - 2012-2013

Thursday 1 November 2012, 14:00 - 17:00

READ THIS FIRST:

- Je mag zelf weten of je de antwoorden in het **Nederlands** of **Engels** opschrijft.
- Clearly write your name and study number on each answer sheet that you use.
- Start each question (number 1, 2, etc.) on a new answer sheet.
- On the first answer sheet, write the total number of answer sheets that you turn in.
- When turning in your answers, please stack your answer sheets in the proper order, and **staple** them together.
- The exam has several questions, it continues on the backside of the papers.
- Note that the lower half of this page lists some useful formulas and constants.
- The exam is open book with limits. You are allowed to use the book by Griffiths or Liboff, the handouts *Extra note on two-level systems and exchange degeneracy for identical particles* and *Feynman Lectures chapter III-1*, and one A4 sheet with your own notes, but nothing more than this.
- If it says "make a rough estimate", there is no need to make a detailed calculation, and making a simple estimate is good enough. If it says "calculate" or "derive", you are supposed to present a full analytical calculation.
- If you get stuck on some part of a problem for a long time, it may be wise to skip it and try the next part of a problem first.
- If you are ready with the exam, please fill in the **course-evaluation question sheet**. You can keep working on the exam until the scheduled end time, and fill it in shortly after that if you like.

NOTE for students that studied the Liboff book and the contents of the course from the year 2010-2011 or earlier:
You should make from this exam sheet only problems 1 and 2, and then the problem L3 from a different handout.
Please walk to the exam supervisor (some 10 min after the start of the exam) and ask for this handout with the replacement for problem 3.
For problem 1, note that spin with $s = ½$ is simply an angular momentum with $j = ½$ or $l = ½$.
Matrix representations for $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ are listed at Eq. (11.78) on p. 514 in the Liboff book.

Useful formulas and constants:

Electron mass	m_e	$= 9.1 \cdot 10^{-31}$ kg
Electron charge	$-e$	$= -1.6 \cdot 10^{-19}$ C
Planck's constant	h	$= 6.626 \cdot 10^{-34}$ Js $= 4.136 \cdot 10^{-15}$ eVs
Planck's reduced constant	$\hbar$	$= 1.055 \cdot 10^{-34}$ Js $= 6.582 \cdot 10^{-16}$ eVs

Fourier relation between x-representation and k-representation of a state

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{\Psi}(k) e^{ikx}\, dk$$

$$\overline{\Psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x) e^{-ikx}\, dx$$

Problem 1

Most large hospitals now have a system for *Magnetic Resonance Imaging* (MRI, Nobel prize 2003). This is an apparatus that can make images of soft tissues inside the human body in a very noninvasive manner. The imaging technique is based on resonantly driving and detecting the signals of the quantum dynamics of nuclear spins. It mainly uses the nuclear spins of hydrogen atoms. Contrast in the images appears because the spin dynamics of the hydrogen nucleus depends on the chemical environment of the hydrogen atom (very small but detectable deviations). One can in this way basically see which organic molecules (that contain hydrogen) are present at a certain place in the body, and also how much water molecules are present at a certain place. The same technique in a physics lab is not called MRI, but Nuclear Magnetic Resonance (NMR).

A person in an MRI machine is brought to a place with a strong magnetic field. Assume for this problem that this is a field of strength $B_Z = 2$ Tesla (typical value in practice, this is 40,000 times stronger than the Earth magnetic field). The field is applied in the z-direction. The spin of the hydrogen nucleus has $s = ½$. The Hamiltonian for this spin is now

$$\hat{H} = -\gamma B_z \hat{S}_z,$$

where is $\gamma = +267.5 \cdot 10^6$ rad s^{-1} T^{-1}, and $\hat{S}_z$ is the operator for the z-component of the spin. The operators for the x- and y-component of the spin are $\hat{S}_x$ and $\hat{S}_y$, respectively. For notation use that the states $|\uparrow\rangle$ and $|\downarrow\rangle$ represent spin-up and spin-down along the z-axis.

a) [1 point]

Which of the operators $\hat{S}_z$, $\hat{S}_x$ and $\hat{S}_y$ commute with the Hamiltonian, and which ones do not? Explain your answer.

b) [1 point]

What are the energy eigenstates and energy eigenvalues of this Hamiltonian? In your answer, also specify which state is the ground state and which state or states is/are excited state(s).

c) [1 point]

The MRI technique relies on applying an oscillating magnetic field that is resonant with the dynamics of the spin. At what oscillation frequency should one apply such an oscillating magnetic field in the MRI setup that we consider here? Calculate an actual number.

d) [2 points]

Calculate (or simply list the answer if you know it) all matrix elements $\langle\uparrow|\hat{S}_j|\uparrow\rangle$, $\langle\uparrow|\hat{S}_j|\downarrow\rangle$, $\langle\downarrow|\hat{S}_j|\uparrow\rangle$ and $\langle\downarrow|\hat{S}_j|\downarrow\rangle$, for the three cases $j = x,y,z$.

e) [3 points]

The spins of the hydrogen nuclei are at time $t = 0$ prepared in the state $|\Psi_0\rangle = \sqrt{\tfrac{2}{3}}|\uparrow\rangle + i\sqrt{\tfrac{1}{3}}|\downarrow\rangle$. Calculate how the expectation values $\langle\hat{S}_x\rangle$, $\langle\hat{S}_y\rangle$ and $\langle\hat{S}_z\rangle$ depend on time for $t > 0$.

Hint: for short notation you could use the results $\langle\uparrow|\hat{S}_j|\uparrow\rangle$ etc. of question d).

f) [2 point]
As follow up on question e),Make graphs for $\langle \hat{S}_x \rangle$, $\langle \hat{S}_y \rangle$ and $\langle \hat{S}_z \rangle$ as a function of time for $t \geq 0$ (so, you must make 3 graphs). Describe in words what the dynamics is of the spins in question e). If you like you can make a drawing with your explanation.

g) [1 point]
If you want to measure the time-dependent spin dynamics of such spins (for example the dynamics of question e) and f)), how could you measure that? Explain in some detail (10 lines of text) what the physical signal is that you would try to detect, and what apparatus you could build for that.

Problem 2
Consider a one-dimensional quantum particle that can move along one direction (x-axis). It does not feel any potentials, $V(x) = 0$ for all x. So, it is a free particle. At some time $t = t_0$, it is in the following state (described as a function of wavenumber k):

$$\overline{\Psi}_0(k) = \begin{cases} 1/\sqrt{2q} & \text{for } |k| < q \\ 0 & \text{for all other } k \end{cases}$$

Here q has the value $q = 1 \cdot 10^{10}$ m^{-1}. The mass m of the particle is $1.055 \cdot 10^{-28}$ kg.

a) [2 points]
Calculate for $t = t_0$ the wavefunction for position, $\Psi(x)$.
Also make a graph (or sketch) of $\Psi(x)$.

b) [1 point]
At time $t = t_0$, you measure the velocity of the particle. Calculate what the probability is to find an answer between $v_1 = 8000$ m s^{-1} and $v_2 = 10000$ m s^{-1}.

c) [2 points]
Immediately after first preparing the particle once more in the state $\overline{\Psi}_0(k)$, you measure the position of the particle. <u>*Estimate*</u> the probability that the measurement gives an outcome between $x_1 = 1\times0.314$ nm and $x_2 = 4\times0.314$ nm. The measurement apparatus has a resolution of 0.001 nm.

d) [2 points]
Directly after the measurement of question c), you measure the velocity of the particle again in the same way as for question b). Make an <u>*estimate*</u> for the probability for finding an answer between $v_1 = 8000$ m s^{-1} and $v_2 = 10000$ m s^{-1} (same interval as for question b)).
(If you cannot make an estimate, describe qualitatively whether the probability is higher, the same or lower as compared to the case of question b).)

e) [1 point]
You prepare the particle once more in the state $\overline{\Psi}_0(k)$, and you are finished doing so at time $t = 0$. Write down an expression that describes this state as a function of time $t > 0$ in k-representation. That is, describe the state as a function of k and t, and work out any operators as a function of k and t as far as possible.

f) [1 point]
Describe in words how the state $\overline{\Psi}(k)$ (the state described in *k*-representation) of the particle of question e) is changing as a function of time for $t > 0$. Include in your answer a remark on how the probability for finding an answer between v_1 = 8000 m s^{-1} and v_2 = 10000 m s^{-1} is changing as a function of time (for when you would measure velocity, same interval as for question b)).

g) [1 point]
Describe in words how the state $\Psi(x)$ (the state described in *x*-representation) of the particle of question e) is changing as a function of time for $t > 0$. Include in your answer a remark on how the probability for finding particle between x_1 = 1×0.314 nm and x_2 = 4×0.314 nm is changing as a function of time (for when you would measure position, same interval as for question c)).

Problem 3
Consider a small box that contains two atoms (and nothing else). You can assume that there are inside the box no magnetic or electric fields. One atom (atom 1) is in a state that has total angular momentum $\boldsymbol{J}_1$. The other atom (atom 2) is in a state that has total angular momentum $\boldsymbol{J}_2$. The length of $\boldsymbol{J}_1$ is $|\boldsymbol{J}_1| = \sqrt{63/4}\,\hbar$, and the length of $\boldsymbol{J}_2$ is $|\boldsymbol{J}_2| = \sqrt{2}\,\hbar$.

a) [2 points]
Can you make a statement about the questions whether atom 1 is a fermion or a boson? And what about atom 2? If you can make such a statement, please work it out. If you cannot make such a statement, explain why not.

b) [2 points]
We now bring the box with the two atoms in an apparatus that measures the length of the vector that represents the total amount of angular momentum in the box. What are the possible measurement outcomes?

c) [2 points]
Assume that the measurement of question b) gave a certain measurement outcome. For a next measurement, the apparatus can be tuned to measure the *z*-component or the *x*-component of the vector that represents the total amount of angular momentum in the box. Discuss what measurement outcomes are now possible. Discuss it both for the case that you measure the *z*-component and the case that you measure the *x*-component.
Note: for the answer consider all possible measurement outcomes of question b). If you do not have an answer for b), assume that the answer on b) is $\sqrt{30}\,\hbar$ [(but also note that this is not the right answer for b)].